IIT-JEE 2010 Mathematics Paper II

PART II - Mathematics

SECTION - I (Single Correct Choice Type)

This section contains 6 multiple choice questions. Each question has four choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

20. Let $S = \{1, 2, 3, 4\}$. The total number of unordered pairs of disjoint subsets of S is equal to
(A) 25 (B) 34 (C) 42 (D) 41

20. **(D)** Each element in either (A) or (B) or neither.
$\therefore$ Total ways $= 3^4 = 81$
$A = B$ iff $A = B = \phi$ (1 case)
otherwise A & B are interchangeable $\therefore n = 1 + (81 - 1) / 2 = 41$

21. Let f be a real valued function defined on the interval $(-1, 1)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1}\, dt$, for all $x \in (-1, 1)$, and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to
(A) 1 (B) 1 / 3 (C) 1 / 2 (D) 1 / e

21. **(B)** Let $f^{-1}(x) = g(x) \Rightarrow f(g)(x)) = x \Rightarrow f'(g(x)).\, g'(x) = 1$

$\therefore g'(2) = \dfrac{1}{f'(g(2))}$, $g(2) = f^{-1}(2) = 0$ as $f(0) = 2$

$\therefore g'(2) = \dfrac{1}{f'(0)}$. Differentiating given equation, we get

$e^{-x}[f'(x) - f(x)] = \sqrt{(x^4 + 1)} \Rightarrow f'(0) - 2 = 1 \Rightarrow f'(0) = 3$
$\therefore$ Required expression $= g'(2) = 1/3$.

22. For $r = 0, 1, \ldots, 10$, let A_r, B_r and C_r denote respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$ is equal to
(A) $B_{10} - C_{10}$ (B) $A_{10}(B^2_{10} - C_{10}A_{10})$ (C) 0 (D) $C_{10} - B_{10}$

22. **(D)** Given expression $= \sum_{r=1}^{10} {}^{10}C_r ({}^{20}C_{10} \cdot {}^{20}C_r - {}^{30}C_{10} \cdot {}^{10}C_r)$

$$= {}^{20}C_{10} \sum_{r=1}^{10} {}^{10}C_r \cdot {}^{20}C_r - {}^{30}C_{10} \sum_{r=1}^{10} ({}^{10}C_r)^2 = {}^{20}C_{10}[{}^{30}C_{20} - 1] - {}^{30}C_{10}[{}^{20}C_{10} - 1]$$
$= {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$.

23. Two adjacent sides of a parallelogram $ABCD$ are given by $\overrightarrow{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$.
The side AD is rotated by an acute angle α in the plane of the parallelogram that AD becomes AD'. If AD' makes a right angle with the side AB, then cosine of the angle α is given by
(A) 8 / 9 (B) $\sqrt{17}$ / 9 (C) 1 / 9 (D) $4\sqrt{5}$ / 9

23. **(B)** $\overrightarrow{AB} \cdot \overrightarrow{AD} = -2 + 20 + 22 = 40 \quad \Rightarrow \quad |\overrightarrow{AB}| \cdot |\overrightarrow{AD}| \cos\theta = 40$

$\Rightarrow \quad \sqrt{1+4+4}\ \sqrt{4+100+121}\cos\theta = 40$

$\Rightarrow \quad 3 \times 15 \cos\theta = 40 \quad \Rightarrow \quad \cos\theta = 40 / 45 = 8 / 9$

AD' makes an angle $(\theta + \alpha)$ with AB

$\overrightarrow{AB} \cdot \overrightarrow{AD'} = 0$.

$\Rightarrow \quad 3 \times 15 \cos(\theta + \alpha) = 0 \quad \Rightarrow \quad \cos(\theta + \alpha) = 0 = \cos(\pi / 2)$

$\Rightarrow \quad \theta + \alpha = \pi / 2 \quad \Rightarrow \quad \theta = (\pi / 2) - \alpha \quad \Rightarrow \quad \cos\theta = \cos((\pi / 2) - \alpha)$

$\Rightarrow \quad 8 / 9 = \sin\alpha \quad \Rightarrow \quad \cos\alpha = \sqrt{17} / 9$.

24. A signal which can be green or red with probability 4 / 5 or 1 / 5 respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is 3 / 4. If the signal received at station B is green, then the probability that the original signal was green is
(A) 3 / 5 (B) 6 / 7 (C) 20 / 23 (D) 9 / 20

24. **(C)** $P = \dfrac{(4/5)\times(3/4)\times(3/4) + (4/5)\times(1/5)\times(1/5)}{(4/5)\times(3/4)\times(3/4) + (1/5)\times(1/4)\times(3/4) + (1/5)\times(3/4)\times(1/4) + (4/5)\times(1/5)\times(1/5)}$

= 20 / 23

25. If the distance of the point $P(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of perpendicular from P to the plane is
(A) (8 / 3, 4 / 3, – 7 / 3) (B) (4 / 3, – 4 / 3, 1 / 3)
(C) (1 / 3, 2 / 3, 10 / 3) (D) (2 / 3, – 1 / 3, 5 / 2)

25. **(A)** $\left|\dfrac{1-4-2-\alpha}{3}\right| = 5 \quad \Rightarrow \quad -5 - \alpha = \pm 15 \quad \Rightarrow \quad \alpha = 10$

Now, $\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z-1}{-2} = r$

Foot of perpendicular $[(r + 1), (2r - 2), (-2r + 1)]$ lies on $x + 2y - 2z - 10 = 0$

$\Rightarrow \quad r + 1 + 4r - 4 + 4r - 2 = 10 \quad \Rightarrow \quad r = 5 / 3$

$\therefore$ Foot of perpendicular = (8 / 3, 4 / 3, – 7 / 3).

SECTION - II (Integer Type)

This section contains 5 questions. The answer to each question is a single-digit-integer, ranging from 0 to 9. The correct digit below the question no. in the ORS is to be bubbled.

26. Let $a_1, a_2, a_3, \ldots\ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$. If $\dfrac{a_1^2 + a_2^2 + \ldots.. + a_{11}^2}{11} = 90$, then the value of $\dfrac{a_1 + a_2 + \ldots.. + a_{11}}{11}$ is equal to

26. **0** Since, $a^2 + (a + d)^2 + \ldots\ldots + (a + 10d)^2 = 990$

$\Rightarrow 11.\ 225 + d^2 .\ 5\ .\ 11.7 + 30\ .\ d.\ 55 = 990$

$\Rightarrow 7d^2 + 30\,d + 27 = 0 \Rightarrow d = -3, -9/7$, of which $d = -3$ is acceptable.

$\therefore$ Expression $= 1/11\ [11/2 \times [15 + 15 + 10\,(-3)] = 0$.

27. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det (adj A) + det (adj B) = 10^6, then [k] is equal to

[**Note:** adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k]

27. **4** det (A) = $8k^3 + 12\,k^2 + 6k + 1 = (2k + 1)^3$ & matrix B is Skew-symmetric matrix, so det.(B) = 0

$\Rightarrow (2k + 1)^6 = 10^6 \Rightarrow k = 9/2 \Rightarrow [k] = 4$.

28. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to

28. **3**. Area = (1 / 2) $ab \sin C$ $\Rightarrow$ $15\sqrt{3} = 1/2\ .\ 6\ .\ 10\ .\ \sin C$

$\Rightarrow$ $\sin C = \sqrt{3}/2$ $\therefore$ $C = 2\pi/3$

$\cos C = (a^2 + b^2 - c^2)/2ab$ $\Rightarrow$ $c = 14$ $\therefore$ $s = 15$

$\therefore$ $r^2 = \Delta^2 / s^2 = 3$.

29. Let f be a function defined on $\boldsymbol{R}$ (the set of all real numbers) such that

$f(x) = 2010\,(x - 2010)\,(x - 2010)^2\,(x - 2010)^3\,(x - 2010)^4$, for all $x \in \boldsymbol{R}$.

If g is a function defined on $\boldsymbol{R}$ with values in the interval $(0, \infty)$ such that

$f(x) = ln\,(g(x))$, for all $x \in \boldsymbol{R}$, then the number of points in $\boldsymbol{R}$ at which g has a local maximum is

29. **1**. $g'(x) / g(x) = 2010\,(x - 2009)\,(x - 2010)^2\,(x - 2011)^3\,(x - 2012)^4$

For maximum, $g'(x)$ must change its sign from positive to negative which is true only at $x = 2009$.

30. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ apart. If the chords subtend at the center, angles of π/k and $2\pi/k$, where $k > 0$, then the value of [k] is

[**Note:** [k] denotes the largest integer less than or equal to k]

30. **3** Let $(\pi/k) = 2\theta \Rightarrow 2\cos\theta + 2\cos 2\theta = \sqrt{3} + 1$ then by observation $\theta = \pi/6 \Rightarrow k = 3$

SECTION - III (Paragraph Type)

This Section contains 2 paragraphs.Based upon the first paragraph 3 multiple Choice questions and based upon the second paragraph 3 multiple choice questions have to be answered. Each of these questions has four choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

Paragraph for questions 31 to 33.

Consider the polynomial

$f(x) = 1 + 2x + 3x^2 + 4x^3$

Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$

31. The real number s lies in the interval

(A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-11, \frac{3}{4}\right)$ (C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

32. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

(A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (C) (9, 10) (D) $\left(0, \frac{21}{64}\right)$

33. The function $f'(x)$ is

(A) increasing in $\left(-t, \frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(B) decreasing in $\left(-t, \frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(C) increasing in $(-t, t)$

(D) decreasing in $(-t, t)$

31-33. 31. **(C)** 32. **(A)** 33. **(B)**

Since $f'(x) = 2(6x^2 + 3x + 1) > 0 \ \forall x \in R \Rightarrow$ only one real root for $f(x) = 0$

Also, $f(0) = 1$, $f(-1) = -2 \Rightarrow$ root must lie in (–1, 0)

Taking average of 0 & –1 $\Rightarrow f(-½) = 1/4 \Rightarrow$ root must lie in (–1, –½)

Similarly, $f(-3/4) = -½ \Rightarrow$ root must lie in (– 3 / 4, – 1 / 2).

Area bounded by the curve, $\int_0^t (1 + 2x + 3x^2 + 4x^3)dx = \frac{t(t^5 - 1)}{(t-1)}$,

with $\frac{1}{2} < t < \frac{3}{4}$.

Let $g(t) = \frac{t(t^5 - 1)}{(t-1)}$, Since, $f''(x) = 6(4x + 1) > 0$

$\Rightarrow g(t)$ must increase as 't' increases. $g(1/2) = 31/32$

$g(3/4) = (525/256)$. $\therefore$ Area must lie in (31/ 32, 525 / 256).

For $f''(x) > 0$ when $x \in (-1/4, t)$ & decreasing in $(-t, -1/4)$.

Paragraph for questions 34 to 36.

Tangents are drawn from the point P (3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

34. The coordinates of A and B are

(A) (3, 0) and (0, 2)

(B) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5},\frac{8}{5}\right)$

(C) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(D) (3, 0) and $\left(-\frac{9}{5},\frac{8}{5}\right)$

35. The orthocenter of the triangle PAB is

(A) $\left(5,\frac{8}{7}\right)$ (B) $\left(\frac{7}{5},\frac{25}{8}\right)$ (C) $\left(\frac{11}{5},\frac{8}{5}\right)$ (D) $\left(\frac{8}{25},\frac{7}{5}\right)$

36. The equation of the locus of the point whose distances from the point P and the line AB are equal, is

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

34 - 36: 34. (D) 35. (C) 36. (A)

$y = mx + \sqrt{(9m^2 + 4)}$ $\Rightarrow$(3,4) should lie on this $\Rightarrow m = \infty, \frac{1}{2}$.

Hence, equation of tangents are $x = 3$ and $2y = x + 5$

$\therefore$ A (3, 0) & B (– (9 / 5), (8 / 5))

For othercentre of ΔPAB : equation of the altitudes

PD and BE are $y = 3x - 5$ and $y = 8/5 \Rightarrow$ orthocentre is $\left(\frac{11}{5},\frac{8}{5}\right)$.

Equation of AB is $3y + x = 3$

Required locus is $\left(\frac{x+3y-3}{\sqrt{10}}\right)^2 = (x-3)^2 + (y-4)^2$

$\Rightarrow 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$.

SECTION - IV (Matrix Type)

This Section contains **2 questions**. Each question has four statements (A,B,C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II.** Any given statements in **Column I** can have correct matching with one or more statement(s) given in **Column II**. For example, if for a given question, statement B matches with the statements given in q and r, then for that particular question, against statement B, darken the bubbles corresponding to q and r in the ORS.

---

37. Match the statements in Column- I with those in Column-II.
[**Note :** Here z takes values in the complex plane and $Im\ z$ and $Re\ z$ denote**,** respectively, the imaginary part and the real part of z.]

Column I	Column II
(A) The set of points z satisfying $\|z - i\|z\|\| = \|z + i\|z\|\|$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\|z + 4\| + \|z - 4\| = 10$	(q) the set of points z satisfying $Im\ z = 0$
(C) If $\|w\| = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(r) the set of points z satisfying $\|Im\ z\| \le 1$
(D) If $\|w\| = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $\|Re\ z\| \le 2$
	(t) the set of points z satisfying $\|z\| \le 3$

37. (A) $\to$ (q, r), (B) $\to$ (p), (C) $\to$ (p,s,t), (D) $\to$ (q,r,s,t)

(A) $|z - i|z|| = |z + i|z||$ $\Rightarrow$ $|x + i(y - \sqrt{(x^2 + y^2)})| = |x + i(y + \sqrt{(x^2 + y^2)})|$
$\Rightarrow$ $4y\sqrt{(x^2 + y^2)} = 0$ $\therefore$ $y = 0$ or $x^2 + y^2 = 0$. $\therefore$ (q, r)

(B) $|z + 4| + |z - 4| = 10$
ellipse with foci (4, 0) and (– 4, 0)
$2a = 10$ and $2ae = 8$ $\therefore$ $e = 4 / 5$.

(C) $w = 2e^{i\theta}$ $\therefore$ $z = w - \frac{1}{w} = 2e^{i\theta} - (1/2)\, e^{-i\theta} = (1/2)\,[3\cos\theta + 5i\sin\theta]$
$2x = 3\cos\theta$, $2y = 5\sin\theta$
$\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow$ $(x^2 / (9/4)) + (y^2 / (25/4)) = 1$
$\therefore$ ellipse with eccentricity 4 / 5
$\therefore$ $|Re\ z| \le 3/2 \le 2$. Also $|z|_{max} = 5/2$ $\therefore$ (p, s, t)

(D) $w = e^{i\theta}$ $\therefore$ $z = 2\cos\theta$ $\therefore$ $Im(z) = 0$ $\therefore$ $|Im(z)| \le 1$
$|Re(z)| = 2\cos\theta \le 2$
$|z| = 2\cos\theta \le 3$ $\therefore$ (q, r , s, t)

38. Match the statements in **Column-I** with the values in **Column -II.**

Column I	Column II
(A) A line from the origin meets the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2}=\frac{y+3}{-1}=\frac{z-1}{1}$ at P and Q respectively. If length $PQ = d$, then d^2 is	(p) – 4
(B) The values of x satisfying $\tan^{-1}(x+3)-\tan^{-1}(x-3)=\sin^{-1}(3/5)$ are	(q) 0
(C) Non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}\cdot\vec{b}=0$, $(\vec{b}-\vec{a})\cdot(\vec{b}+\vec{c})=0$ and $2\lvert\vec{b}+\vec{c}\rvert=\lvert\vec{b}-\vec{a}\rvert$. If $\vec{a}=\mu\vec{b}+4\vec{c}$, then the possible values of μ are	(r) 4
(D) Let f the function on $[-\pi,\pi]$ given by $f(0)=9$ and $f(x)=\sin(9x/2)/\sin(x/2)$ for $x\neq 0$. The value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$ is	(s) 5
	(t) 6

38. (A) → (s), (B) → (p, r), (C) → (q), (D) → (r)

(A) If point $(2 + a, 1 - 2a, -1 + a)$ and $(8/3 + 2b, -3 - b, 1 + b)$, then

$$\frac{2+a}{8/3+2b}=\frac{1-2a}{-3-b}=\frac{-1+a}{1+b}=\frac{a}{2}=\frac{4-a}{2/3}$$

⇒ $a = 3$, $b = 4/15$

∴ Points $(5, -5, 2)$ and $(10/3, -10/3, 4/3)$ giving $d^2 = 6$ ∴ (s)

(B) ∵ $\tan^{-1}(x+3)-\tan^{-1}(x-3)=\tan^{-1}(3/4)$

⇒ $\frac{(x+3)-(x-3)}{1+(x^2-9)}=3/4$ ⇒ $6/(x^2-8)=3/4$

⇒ $24 = 3x^2 - 24$ ⇒ $x = \pm 4$. ∴ (p, r)

(C) $z_b = ki\, z_a$

$z_a + (4-\mu)\, z_b = \pm 2i\,(z_b - z_a)$

$(4-\mu)\,k\,i + 1 = 2i + 2k$ or $-2i - 2k$

$k = 1/2$ and $4-\mu = 4$ or $k = -1/2$ and $4-\mu = 4$

⇒ $\mu = 0$. ∴ (q)

(D) Using $\sin(9x/2) - \sin(7x/2) = 2\sin(x/2)\,4x$

$$\int_{-\pi}^{\pi}\frac{(\sin(9x/2)-\sin(7x/2))dx}{\sin(x/2)}=0$$

∴ $I=\int_{-\pi}^{\pi}\frac{\sin(x/2)}{\sin(x/2)}dx=2\pi$ ∴ Expr. = $(2/\pi)\times 2\pi = 4$. ∴ (r)